

DIFFERENTIAL FORMS FOR FRACTAL SUBSPACES AND FINITE ENERGY COORDINATES

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ABSTRACT. This paper introduces a notion of differential forms on closed, potentially fractal, subsets of the \mathbb{R}^m by defining pointwise cotangent spaces using the restriction of C^1 functions to this set. Aspects of cohomology are developed: it is shown that the differential forms are a Banach algebra and it is possible to integrate these forms along rectifiable paths. These definitions are connected to the theory of differential forms on Dirichlet spaces by considering fractals with finite energy coordinates. In this situation, the C^1 differential forms project onto the space of Dirichlet differential forms. Further, it is shown that if the intrinsic metric of a Dirichlet form is a length space, then the image of any rectifiable path through a finite energy coordinate sequence is also rectifiable. The example of the harmonic Sierpinski gasket is worked out in detail.

1. INTRODUCTION

Considering a closed, potentially fractal, subset of \mathbb{R}^m , we define cotangent spaces as the quotients of C^1 functions on the space. We refer to these as C^1 differentials. While this is a standard construction when considering subvarieties with algebraic, smooth or analytic functions, the current work makes use of functions which have lower regularity: only assuming one continuous derivative, and we take an arbitrary closed subset rather than an affine variety. Theorem 2.9 establishes the equivalence of three definitions of these cotangent spaces. In section 3, C^1 differential forms are defined. The main result of this section is that the 1-dimensional differential forms are a Banach algebra when added to the scalar forms (functions).

In the classical setting, it is natural to define differential 1-forms as field which can be integrated along curves. This is a particular advantage of the present work — Section 4 proves that it is possible to integrate C^1 differential forms along rectifiable curves which remain in the closed subset. [CGIS13] discusses the possibility of defining integrals of differential forms on the Sierpinski gasket along curves. Another method, which considers integration on fractal curves is [Har99].

In section 5, if the subset has a suitable measure, it is shown that the direct integral of the cotangent bundles can be taken to define L^2 differential forms on the space which contain the C^1 forms. Theorem 4.1 uses integration along curves to prove that the exterior derivative is a closed operator with respect to this direct integral structure on these L^2 forms.

Recently there have emerged many points of view and techniques aimed at understanding differential structures on metric measure spaces. Works including [CS07, IRT12] develop differential forms on fractals from energy measures on Dirichlet spaces, which we shall refer to as Dirichlet differential forms. This construction allows for the study of more

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general differential equations on fractal spaces. For example, one can construct magnetic Schrödinger operators, as in [HT14, HR16, HKM⁺16], allowing for the rigorous mathematical study of physical objects from [DABK83, Bel92]. For one-dimensional fractals Hodge theory is defined in [HT14], and [HKT15] defines a Dirac operator and spectral triples with these fractals. Navier-Stokes equations are also studied in [HT14, HRT13].

Section 6 discusses the relationship between the C^1 differential forms and those defined for Dirichlet Spaces which have finite energy coordinates. One of the main result of this section is to prove that there is a closed projection from the C^1 differential forms defined to the Dirichlet differential forms. Subsection 6.1 gives the details of this relationship in the special case of harmonic coordinates on the Sierpinski Gasket as defined in [Kaj12, Kaj13, Kig93, Kig08].

The work in [Gig15] constructs first and second order differential structures for metric spaces which satisfy Ricci curvatures lower bounds. The works [BSSS12, SS12, Sma15], construct abstract versions of Hodge–DeRham and Alexander–Spanier cohomologies for use on metric spaces. A large part of the motivation is to understand data sets by there global structure which is determined by these cohomologies.

There is a strong relationship between geometry of a metric measure space and Dirichlet forms on the space. This is discussed at length in [Stu94, Sto10, HKT12], where intrinsic metrics induced by the Dirichlet space are proven to be geodesic metrics in the sense that the distance is given by the length of the shortest path between two points. Theorem 6.1 of the current work proves that rectifiable curves on our fractal (with respect to the intrinsic metric) have rectifiable (with respect to Euclidean distance) images through our coordinates.

Areas of interest for further research would be extending these results to infinite dimensional spaces. This would allow for the study of Dirichlet forms with infinite coordinate sequences, as was considered in previous works [Hin10]. Further, one could consider sub-Riemannian spaces, as considered in [GL14].

2. DEFINITIONS OF COTANGENT SPACES FOR CLOSED SUBSETS

Consider U be an open subset of \mathbb{R}^m , $C_0(\overline{U})$ shall denote the set of continuous functions on the closure of U vanishing at infinity, and $C^1(U)$ will denote the continuous functions which have continuous first-order partial derivatives. Define the following norm on $C_0^1(U) := C^1(U) \cap C_0(\overline{U})$

$$\|u\|_{C^1} := \|u\|_{\infty} + \sum_{i=1}^m \left\| \frac{\partial u}{\partial x^i} \right\|_{\infty}$$

where

$$\|u\|_{\infty} = \sup_{x \in U} |u(x)|$$

and $\{x^i\}_{i=1}^m$ are the coordinates of \mathbb{R}^m . It is elementary to prove

Proposition 2.1. *$C_0^1(U)$ is a commutative Banach algebra with pointwise addition, multiplication, and the norm $\|\cdot\|_{C^1}$. If U has compact closure, then $C_0^1(U)$ has multiplicative identity 1_U .*

For a subset $K \subset U$ define

$$\mathcal{I}_K = \{u \in C_0^1(U) \mid u(p) = 0 \text{ for all } p \in K\}$$

If K is a relatively closed subset of U , then \mathcal{I}_K is a closed ideal of $C_0^1(U)$. We shall take $\mathcal{I}_p := \mathcal{I}_{\{p\}}$ for a point $p \in U$.

Proposition 2.2. *For a given closed $K \subset U$, The space $C_0^1(K) := C_0^1(U)/\mathcal{I}_K$, is a Banach algebra with the norm*

$$\|u\|_K := \|u\|_{\infty, K} + \inf_{v|_K = u|_K} \sum_{i=1}^d \|\partial_i v\|_{\infty, K}$$

where $\|v\|_{\infty, K} = \sup_{p \in K} |v(p)|$.

Proof. \mathcal{I}_K is a closed ideal of $C_0^1(U)$ with respect to the norm above and this norm is the quotient norm of $C_0^1(U)/\mathcal{I}_K$. \square

Remark 2.3. We interpret this to mean that every function in an equivalence class of $C_0^1(K)$ takes the same values on K , so we tend to think of elements in $C_0^1(K)$ as restrictions of elements in $C_0^1(U)$.

For $f \in C^1(U)$ denote the classical gradient $\nabla f : U \rightarrow \mathbb{R}^m$ as $\nabla f := (\frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^m})$ for all $f \in C^1(U)$.

Proposition 2.4. *If f is a $C_0^1(U)$ with $f(p) = 0$ and $\nabla f(p) = 0$, then there is $g_1 \in C(U)$ and $g_2(p) \in \mathcal{I}_p$ such that $f(x) = g_1(x)g_2(x) + g_0(x)$ where $g_1(p) = g_2(p) = 0$, $g_1 \in C^1(U \setminus \{p\})$, and g_0 is constant in a neighborhood of p . If \overline{U} is compact, we can take $g_0 \equiv 0$.*

Proof. By a partition of unity argument, it follows that f is the sum of a function compactly supported in a neighborhood around p and a function which is 0 in a neighborhood of p . Thus we assume that f is compactly supported around p without losing generality. Further, we may assume that $p = 0$.

Because $\lim_{x \rightarrow p} |f(x)|/|x| = 0$, we can find an increasing $C^1((0, N)) \cap C([0, N))$ function $h(t) > \sup_{|x| \leq t} |f(x)|/|x|$ and $h(0) = 0$. To construct this function we define $h(1/n) = \sup_{x \leq 1/(n-1)} |f(x)|/|x|$ and interpolate with a C^1 function. With this $g_1(x) = (h(|x|))^{1/2}$ and

$$g_2(x) = \begin{cases} f(x)/(h(|x|))^{1/2} & \text{if } x \neq p \\ 0 & \text{if } x = p, \end{cases}$$

which is continuous because

$$\lim_{x \rightarrow p} \frac{f(x)}{(h(|x|))^{1/2}} \leq \lim_{x \rightarrow p} (|x|f(x))^{1/2} = 0$$

Clearly g_1 is in $C(U) \cap C^1(U \setminus \{p\})$, and

$$\lim_{x \rightarrow p} \frac{|f(x)|}{|x|(h(|x|))^{1/2}} \leq \lim_{x \rightarrow p} \left(\frac{|f(x)|}{|x|} \right)^{1/2} = 0.$$

so g_2 is in $C^1(U)$ and vanishes at p . \square

We define

$$\mathcal{I}_{2,p} := \text{clos}(\mathcal{I}_p^2, \|\cdot\|_{C^1}),$$

that is, the closure of the square of the ideal \mathcal{I}_p^2 with respect to the norm $\|\cdot\|_{C^1}$.

Corollary 2.5. *The ideal $\mathcal{I}_{2,p}$ consists the set of functions of the form $g_1 g_2 + g_0$ where $g_0 \in C_1$ is equivalently 0 in a neighborhood of p , and $g_1(p) = g_2(p) = 0$ with $g_1 \in C(U) \cap C^1(U \setminus \{p\})$ and $g_2 \in C_0^1(U)$.*

Definition 2.6. Define $T_p^*U := \mathcal{I}_p / \mathcal{I}_{2,p}$, and \mathcal{K}_p the span of a compactly supported smooth function ϕ which is constant 1 in a neighborhood of p . Then $C_0^1(U) = \mathcal{K}_p \oplus T_p^*U \oplus \mathcal{I}_{2,p}$, and thus T_p^*U is an m -dimensional vector space. The natural projection from $d_p : C^1(U) \rightarrow T_p^*U$ as the exterior derivative. Note that d_p does not depend on our choice of ϕ , because if ψ was another such function, then $\phi - \psi \in \mathcal{I}_{2,p}$.

There is a unique decomposition of f into a sum of elements from \mathcal{K}_p , T_p^*U and $\mathcal{I}_{2,p}$ given by the Taylor expansion

$$f(x) = f(p)\phi(x) + (x - p) \cdot \nabla f(p) + o(|x - p|^2).$$

Proposition 2.7. $d_p(fg) = f(p)d_p g + g(p)d_p f$ and

$$d_p f = \frac{\partial f}{\partial x^i} d_p x^i.$$

Remark 2.8. This is classical, of course, but the following proof exhibits thinking which is useful herein.

Proof. Assuming, without loss of generality, $p = 0$, the direct product decomposition (Taylor's theorem) implies that, if $[f]$ and $[g]$ are the equivalence classes of $f, g \in C_0^1(U) \bmod \mathcal{I}_{2,p}$,

$$[f(x)] = f(p)\phi(x) + x \cdot \nabla f(p) + \mathcal{I}_{2,p} \quad \text{and} \quad [g(x)] = g(p)\phi(x) + x \cdot \nabla g(p) + \mathcal{I}_{2,p}$$

where ϕ is the bump function mentioned above. The multiplying we discover

$$[g(x)f(x)] = f(p)g(p)\phi(x) + \phi(x)(g(p)(x \cdot \nabla f(p)) + f(p)(x \cdot \nabla g(p))) + \mathcal{I}_{2,p}$$

noting that $\phi^2 = \phi$ modulo $\mathcal{I}_{2,p}$ and that ϕ is equivalently 1 in a neighborhood of p . \square

Now, fixing a closed $K \subset U$, we define

$$C_0^1(K) := C_0^1(U) / \mathcal{I}_K$$

Further, we define $\mathcal{I}_p(K) = \mathcal{I}_p / (\mathcal{I}_K \cap \mathcal{I}_p)$, alternatively

$$\mathcal{I}_p(K) = \begin{cases} \mathcal{I}_p / \mathcal{I}_K & \text{if } p \in K \\ \{0\} & \text{if } p \notin K. \end{cases}$$

Note that this is a subspace of $C_0^1(K)$, and is a Banach algebra with the inherited norm. We define

$$\mathcal{I}_{2,p}(K) := \mathcal{I}_{2,p} / \mathcal{I}_K \cap \mathcal{I}_{2,p} \cong (\mathcal{I}_{2,p} + \mathcal{I}_K) / \mathcal{I}_K,$$

noting that both are Banach algebras, because \mathcal{I}_K and $\mathcal{I}_{2,p}$ are both closed, so their intersection is also a closed ideal. This notation allows us to define $T_p^*K = \mathcal{I}_p(K) / \mathcal{I}_{2,p}(K)$. The homomorphism theorems for rings provides the following equivalent definitions.

Theorem 2.9. *Using the notation above,*

$$T_p^*K := \mathcal{I}_p(K)/\mathcal{I}_{2,p}(K) \cong \mathcal{I}_p/(\mathcal{I}_{2,p} + \mathcal{I}_K) \cong T_p^*U/d_p(\mathcal{I}_K).$$

Thus

$$\begin{aligned} C_0^1(U) &= \mathcal{K}_p \oplus T_p^*K \oplus (\mathcal{I}_{2,p} + \mathcal{I}_K) \\ &= \mathcal{K}_p \oplus T_p^*K \oplus \mathcal{I}_{2,p}(K) \oplus \mathcal{I}_K \end{aligned}$$

and thus

$$C_0^1(K) = \mathcal{K}_p \oplus T_p^*K \oplus \mathcal{I}_{2,p}(K).$$

and thus we define the differential $d_p^K : C_0^1(K) \rightarrow T_p^*K$ by the natural projection associated with the above decomposition. If we define $\rho_p : T_p^*U \rightarrow T_p^*K$ and $\sigma : C_0^1(U) \rightarrow C_0^1(K)$ to be the natural projections, then $d_p^K \circ \sigma = \rho_p \circ d_p$, i.e. the following diagram commutes

$$\begin{array}{ccc} C_0^1(U) & \xrightarrow{d_p} & T_p^*U \\ \downarrow \sigma & & \downarrow \rho \\ C_0^1(K) & \xrightarrow{d_p^K} & T_p^*K \end{array}$$

This commutative diagram implies the following.

Proposition 2.10. $d_p^K(fg) = f(p)d_p^Kg + g(p)d_p^Kf$.

3. DEFINITION OF DIFFERENTIAL FORMS

Definition 3.1. Assuming T_p^*U has the standard norm, for $\omega = \sum_{i=1}^m \omega_i dx^i$, then $\|\omega\|_p^2 = \sum_{i=1}^m \omega_i^2(p)$. Since T_p^*K is a quotient space of T_p^*U , then we define the quotient norm for any equivalence class $[\omega] \in T_p^*K$ by

$$\|[\omega]\|_{T_p^*K} = \inf_{\eta \in [\omega]} \sqrt{\sum_{i=1}^m \eta_i^2}, \quad \text{for } \eta = \sum_{i=1}^m \eta_i dx^i.$$

Similarly, assuming that T_p^*U has the standard inner product $\langle \eta_i dx^i, \omega_i dx^i \rangle_p = \sum_{i=1}^m \eta_i \omega_i$, we give T_p^*K the standard quotient inner product: $\langle [\eta], [\omega] \rangle_{K,p} = \langle \tilde{\eta}, \tilde{\omega} \rangle$, where $\tilde{\eta}, \tilde{\omega} \in (d\mathcal{I}_K)^\perp$ are the unique representative of $[\eta], [\omega]$ respectively.

Definition 3.2. On the other hand if T_pU is the tangent space at a point p , define the tangent space with respect to K at a point p

$$T_pK = \{X \in T_pU \mid Xf = 0 \ \forall f \in \mathcal{I}_K\}.$$

T_pK is a subspace of T_pU , and we shall define $P_p : T_pU \rightarrow T_pK$ to be the orthogonal projection (with respect to the dot product). Since the definition is equivalent to being the set of $X \in T_pU$ such that $\omega X = 0$ for all $\omega \in d\mathcal{I}_K$, T_pK is the dual of T_p^*K and visa versa.

If we take $\sharp : T_p^*U \rightarrow T_pU$ to be the standard musical operator such that $\sharp(\sum \omega_i dx^i) = \sum \omega_i \frac{\partial}{\partial x^i}$ then we have that the following diagram commutes

$$\begin{array}{ccc} T_p^*U & \xrightarrow{\sharp} & T_pU \\ \downarrow \rho & & \downarrow P \\ T_p^*K & \xrightarrow{\sharp} & T_pK \end{array}$$

i.e. $P\sharp = \sharp\rho$, and $\|[\omega]\|_{T_p^*K} = \|P\sharp\omega\|$.

Define the cotangent bundle of U to be $T^*U = \coprod_{p \in U} T_p^*U$ and $\Omega_C^1(U)$ to be the continuous sections of T^*U which have fiberwise bounded norms vanishing at infinity, i.e. maps from $U \rightarrow T^*U$ of the form $p \mapsto \sum_{i=1}^m g_i(p) d_p x^i$ for $g_i \in C_0(\overline{U})$, which we will denote $\sum_{i=1}^m g_i dx^i$. Elements of $\Omega_C^1(U)$ can also be interpreted as continuous functions from U to \mathbb{R}^m .

Proposition 3.3. $\Omega_C^1(U)$ is a Banach space with the norm

$$\|\omega\|_{\Omega_C^1} = \sup_{p \in U} \|\omega_p\|_p = \sup_{p \in U} \sqrt{\omega_1^2 + \omega_2^2 + \cdots + \omega_m^2}, \quad \text{for } \omega = \omega_i dx^i.$$

Similarly, define $TU = \coprod_{p \in U} T_pU$ to be the tangent bundle. We shall use \sharp to refer to the musical isomorphism between T^*U and TU .

We shall define the cotangent bundle of K to be $T^*K = \coprod_{p \in U} T_p^*K$, and $\Omega_C^1(K)$ to be the sections of T^*K which are of the form $p \mapsto \rho_p \omega_p$ where $\omega \in \Omega_C^1(U)$. That is we shall consider the maps $p \mapsto \sum_i g_i(p) d_p^K \tilde{x}^i$, where $g_i \in C(K)$, and \tilde{x}^i is the equivalence class of x^i in $C_0^1(K)$. Two forms ω and η are equal, if $\omega_p - \eta_p$ are in $d\mathcal{I}_p$.

Define $\rho : \Omega_C^1(U) \rightarrow \Omega_C^1(K)$ to be defined fibre-wise as the projection $\rho_p : T_p^*U \rightarrow T_p^*K$. Since convergence with respect to the sup norm on $\Omega_C^1(U)$ implies pointwise convergence, it follows that that $\ker \rho$ is a closed subspace of $\Omega_C^1(U)$.

We can define the norm on $\Omega_C^1(K)$ by

$$\|\omega\|_{\Omega_C^1(K)} = \inf_{\eta \sim \omega} \|\eta\|_{\Omega_C^1(U)}.$$

Proposition 3.4. $\Omega_C^1(K)$ is a Banach space with the norm $\|\cdot\|_{\Omega_C^1(K)}$.

Proof. First, we claim that this is a well defined norm on $\Omega_C^1(K)$, and in particular that $\|\omega\|_{\Omega_C^1(K)} = 0$ if and only if $\omega_p \in d\mathcal{I}_{p,2}$ for all $p \in K$. If $\|\omega\|_{\Omega_C^1(K)} = 0$ this implies that $\omega \sim 0$ and thus $\omega_p \in d\mathcal{I}_{p,2}$ for all $p \in K$.

On the other hand, let $\omega = \sum \omega_i(p) d_p^K x^i$ is such that $\omega_p \in d\mathcal{I}_{p,2}$. Because for $\omega_i \in C_0(K)$ it can be extended to a function in $\tilde{\omega}^i \in C_0(U)$, and defining $\tilde{\omega} = \sum \tilde{\omega}^i(p) dx^i$ is equivalent to 0. Hence $\|\omega\|_{\Omega_C^1(K)} = 0$. \square

On the other hand define

$$\Omega_1^C(K) = \{PX \mid X \in \Omega_1^C(U)\}$$

where $\Omega_1^C(U)$ is the set of continuous vector fields on U , i.e. $X = \sum X^i \frac{\partial}{\partial x^i}$ where $X^i \in C_0^1(\overline{U})$, and $(PX)_p = P_p X_p$. We can consider \sharp as a map from $\Omega_C^1(K)$ to $\Omega_1^C(K)$,

and this allows for the characterization

$$\|\omega\|_{\Omega_C^1} = \sup_p \|P^\sharp \omega\|.$$

Proposition 3.5. *With fiberwise multiplication, $C_0(K)$ acts on $\Omega_C^1(K)$ by bounded operators. In particular $\|f\omega\|_{\Omega_C^1} \leq \|f\|_\infty \|\omega\|_{\Omega_C^1}$. Similarly, if $f \in C_0^1(K)$, then defining $f\omega = \tilde{f}\omega$ to be fiberwise multiplication by any representative $\tilde{f} \in C_0^1(U)$ of f , then $\|f\omega\|_{\Omega_C^1} \leq \|f\|_{C^1} \|\omega\|_{\Omega_C^1}$.*

Now, if we define $\Omega(U) = C_0^1(U) \oplus \Omega_C^1(U)$ then we have can define a multiplication

$$(f_1, \omega_1)(f_2, \omega_2) = (f_1 f_2, f_1 \omega_2 + f_2 \omega_1).$$

Theorem 3.6. $\Omega(U)$ is a Banach algebra with the norm

$$\|(f, \omega)\|_\Omega = \|f\|_{C^1} + \|\omega\|_{\Omega_C^1}.$$

Proof. $\Omega(U)$ is a Banach space because $C_0^1(U)$ and $\Omega_C^1(U)$ are both Banach spaces with their norms. To see the algebra condition

$$\begin{aligned} \|(f_1, \omega_1)(f_2, \omega_2)\|_\Omega &= \|f_1 f_2\|_{C^1} + \|f_1 \omega_2 + f_2 \omega_1\|_{\Omega_C^1} \\ &\leq \|f_1\|_{C^1} \|f_2\|_{C^1} + \|f_1\|_{C^1} \|\omega_2\|_{\Omega_C^1} + \|f_2\|_{C^1} \|\omega_1\|_{\Omega_C^1} \\ &\leq \|(f_1, \omega_1)\|_\Omega \|(f_2, \omega_2)\|_\Omega. \end{aligned}$$

□

4. HOMOLOGY AND COHOMOLOGY

It is important to note that if $\gamma = (\gamma_1, \dots, \gamma_m) : [a, b] \rightarrow U$ is a curve which is differentiable at t_0 , then γ induces an element of $\dot{\gamma}(t_0) \in T_p U$, where $\gamma(t_0) = p$ in a standard way

$$\dot{\gamma}(t_0)f = \frac{d}{dt}(f \circ \gamma)(t_0) = \left(\frac{d\gamma_1}{dt}, \dots, \frac{d\gamma_m}{dt} \right) \cdot \nabla f.$$

Proposition 4.1. *Say $\gamma : [a, b] \rightarrow K$ is a continuous curve which is differentiable at time $t = t_0$ such that $\gamma(t_0) = p \in K$, then $\dot{\gamma}(t_0) \in T_p K$, i.e. if f and g are elements of $C_0^1(K)$ such that $d_p^K f = d_p^K g$, then*

$$\frac{d}{dt}f \circ \gamma(t_0) = \frac{d}{dt}g \circ \gamma(t_0).$$

Proof. Assuming $f - g \in \mathcal{I}_{p,2}(K)$, then for any representatives from $C_0^1(U)$, \tilde{f}, \tilde{g} respectively, $\tilde{f} - \tilde{g} \in \text{span}(\mathcal{I}_{2,p} + \mathcal{I}_K)$, that is $\tilde{f} - \tilde{g} = \phi + \psi$ where $\phi \in \mathcal{I}_{2,p}$ and ψ is constant on K , thus $\nabla(\tilde{f} - \tilde{g}) = \nabla\psi$ and

$$\frac{d}{dt}(f - g) \circ \gamma(t) = \dot{\gamma}(t)(\tilde{f} - \tilde{g}) = \dot{\gamma}(t)\psi = \frac{d}{dt}\psi \circ \gamma(t_0) = 0.$$

□

This fact allows us to integrate differential forms in T^*K along paths which stay in K .

Theorem 4.1. For a rectifiable curve $\gamma : [a, b] \rightarrow K$, the linear functional from $\Omega_C^1(K) \rightarrow \mathbb{R}$

$$\omega \mapsto \int_{\gamma} \omega := \int_{\gamma} \eta$$

where $\eta \in \Omega_C^1(U)$ is fiber-wise a representative of ω , is well defined.

Proof. First, we know that we can chose such an η by the definition of $\Omega_C^1(K)$. Say η and η° are two representatives of ω , as proven in the proposition above $\dot{\gamma}(t)(\eta - \eta^\circ) = 0$ for all t , thus

$$\int_{\gamma} \eta = \int_a^b \dot{\gamma}(t) \eta \, dt = \int_a^b \dot{\gamma}(t) \eta^\circ \, dt = \int_{\gamma} \eta^\circ.$$

□

This implies a version of the fundamental theorem of line integrals for the restricted cotangent space.

Theorem 4.2. For any rectifiable curve $\gamma : [a, b] \rightarrow K$ and any function $f \in C_0^1(K)$, $\int_{\gamma} d^K f = f(b) - f(a)$.

Theorem 4.3. If for every two points in $x, y \in K$ there is a rectifiable curve $\gamma : [a, b] \rightarrow K$ such that $\gamma(a) = x$ and $\gamma(b) = y$, then $d^K : C_0^1(K) \rightarrow \Omega_C^1(K)$, is closed as an operator from $C_0(K) \rightarrow \Omega_C^1(K)$, where $C_0(K)$ is the continuous functions vanishing at infinity with the uniform norm.

Proof. Say that $f_i \rightarrow f$ and $\omega_i \rightarrow \omega$, for $f_i, f \in C_0^1(K)$ and $\omega_i, \omega \in \Omega_C^1(K)$ with $d^K f_i = \omega_i$. Pick $x_0 \in K$, then, for each $x \in K$ pick $\gamma_x : [a, b] \rightarrow K$ with $\gamma_x(a) = x_0$ and $\gamma_x(b) = x$, then we can define

$$g(x) = f(x_0) + \int_{\gamma_x} \omega.$$

Note that by the theorem 4.2, this function is independent of our choice of γ_x . It is also clear that

$$f_i(x) = f_i(x_0) + \int_{\gamma_x} \omega_i$$

by the theorem 4.2. Finally, because $\dot{\gamma}(t)\omega_i \rightarrow \dot{\gamma}(t)\omega$ uniformly, we have that $f = g$. □

5. IF K IS A METRIC MEASURE SPACE

In this section, concepts from measurable spaces are introduced to help us understand the underlying space from an intrinsic viewpoint. Assume that K is endowed with σ -finite Borel regular measure μ .

The subset $TK = \coprod_{p \in K} T_p K$ is closed in TU because if $(x_n, X_n) \rightarrow (x, X)$ in TU and $X_n f = 0$ for all $f \in \mathcal{I}_K$, then $X f = 0$. Consider $\langle \cdot, \cdot \rangle_{K,p}$ to be the quotient inner product induced by the the standard metric on $T_p^* U$, and $\|\cdot\|_{K,p}$ be the associated norm. If $P_p : T_p U \rightarrow T_p K$ is taken to be the orthogonal projection onto the tangent space, then if $X_i = \frac{\partial}{\partial x^i}$ is the standard frame for TU , then $\|dx^i\|_{K,p} = \|P_p X_i\|$.

Lemma 5.1. The values $\langle d^K x^j, d^K x^i \rangle_{K,p}$ are measurable on K and hence $T_x^* K$ is a measurable field.

Proof. If $p_k \rightarrow p$, then $\|d_p^K x^i\|_{K,p} \geq \limsup_k \|d_{p_k}^K x^i\|_{K,p}$. This follows because, taking $T_p K$ to be the tangent space at p , $P_{p_k} : T_{p_k} U \rightarrow T_{p_k} K$ to be the orthogonal projection, this means that $\|d_p^K x^i\|_{K,p} = \|P_p X^i\|$.

If we restrict to a subsequence such that $\|d_{p_k}^K x^i\|_{K,p}$ converges to something greater than 0 (the claim is trivial if such a subsequence does not exist). Then, because the sequence is eventually contained in a compact neighborhood of $(p, 0) \in TU$, there is a further subsequence that $Y_k = P_{p_k} X_i$ converges to $Y \in T_p U$. Since the norm is continuous,

$$Y = \lim_{k \rightarrow \infty} \frac{\langle Y_k, X_i \rangle}{\|Y_k\|^2} Y_k = \lim_{k \rightarrow \infty} \frac{\langle Y, X_i \rangle}{\|Y\|^2} Y$$

but since Y is in $T_p K$ this implies that

$$\|P_p X_i\| \geq \frac{\langle Y, X_i \rangle}{\|Y\|} = \lim_{k \rightarrow \infty} \|d_{p_k}^K x^i\|_{K,p},$$

□

Since, $T_p K$ is a measurable field over K , it is possible to consider the direct integral of this field, as follows.

Definition 5.2. For the rest of this section we shall assume that μ is a Radon measure on U with support in K . We define measurable forms on K by the direct integral

$$\Omega_{L^2}^1(K, \mu) = \int_U^\oplus T_x^* K \, d\mu(x).$$

We shall write $\Omega_{L^2}^1(K)$ when the choice of measure is clear. This is also a Hilbert space with

$$\langle \omega, \eta \rangle_K = \int_K \langle \omega_p, \eta_p \rangle_{K,p} \, d\mu(p).$$

We consider $\Omega_C^1(K) \subset \Omega_{L^2}^1(K)$ in the natural way.

Theorem 5.3. $\Omega_C^1(K)$ is a dense subspace of $\Omega_{L^2}^1(K)$.

Proof. Every element in $\Omega_{L^2}^1(K)$ is the space of measurable sections $\sum_{i=1}^m \omega_i d^K x^i$ for $\omega_i \in L^2(K, \mu)$. In this case

$$\|\omega\|_{\omega_{L^2}(K, \mu)} = \sum_{i,j=1}^m \int \omega_i \omega_j \langle d^K x^i, d^K x^j \rangle \, d\mu$$

By approximating each ω_i by continuous functions with respect to μ and noting that $|\langle d^K x^i, d^K x^j \rangle| \leq 1$, one sees that ω can be approximated by a forms with these coefficients. □

If ν is another Radon measure with support on K with $\nu \ll \mu$, then there is a natural map $\Omega_{L^2}^1(K, \nu) \rightarrow \Omega_{L^2}^1(K, \mu)$ by $\omega \mapsto (d\mu/d\nu) \omega$.

6. TWO NOTIONS OF 1-FORMS

In this section we shall compare the differential forms from [IRT12, CS07, HRT13] to C^1 differential forms. First we shall recall some basic definitions and concepts from the theory of Dirichlet forms, for more see [BH91, FOT11]. For a locally compact metric space K , consider a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$, where μ is a Radon measure. That is $(\mathcal{E}, \mathcal{F})$ satisfy

(DF1) $\mathcal{F} \subset L^2(K)$ is a dense subspace and $\mathcal{E} : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}$ is a non-negative definite symmetric bilinear form.

(DF2) *Closed*: \mathcal{F} is a Hilbert space with the inner product

$$\mathcal{E}_1(f, g) := \mathcal{E}(f, g) + \langle f, g \rangle_{L^1(K, \mu)}.$$

(DF3) *Markov property*: $f \in \mathcal{F}$ implies that $\hat{f} = (0 \vee f) \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(\hat{f}, \hat{f}) \leq \mathcal{E}(f, f)$.

(DF4) *Regular*: The space $\mathcal{C} := C_c(K) \cap \mathcal{F}$ is uniformly dense in $C_c(K)$ and dense in \mathcal{F} with respect to the norm induced by \mathcal{E}_1 .

It shall also be assumed that $(\mathcal{E}, \mathcal{F})$ is strongly local: For $u, v \in \mathcal{F}$, if u is constant in the support of v , then $\mathcal{E}(u, v) = 0$.

From [BH91, Chapter I 3.3], the space \mathcal{C} from (DF4) is an algebra with respect to pointwise multiplication and addition, thus we shall refer to it as the Dirichlet algebra. For any $f \in \mathcal{C}$ define the energy measure

$$\int \phi d\Gamma(f) = \mathcal{E}(\phi f, f) - \frac{1}{2} \mathcal{E}(\phi, f^2) \text{ when } \phi \in \mathcal{C}.$$

Consider the space $\mathcal{C} \otimes \mathcal{C}$ with the bilinear form

$$\langle a \otimes b, c \otimes d \rangle_{\mathcal{H}} = \int_K bd \, d\Gamma(a, c).$$

As in [HRT13], define the differential forms on the space K to be the space \mathcal{H} , which is attained from $\mathcal{C} \otimes \mathcal{C}$ by factoring out the zero space of $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and completing.

Here we shall assume, that there is a finite coordinate sequence $\Phi = (\phi^i)_{i=1}^m$ of finite energy functions $\phi^i : K \rightarrow \mathbb{R}$, that is

(CO1) for all $i, j \in \mathbb{N}$, $d\Gamma(\phi^i, \phi^j)/d\mu \in L_1(K, \mu) \cap L_\infty(K, \mu)$,

(CO2) the space of functions

$$C^1(\Phi) = \{F(\phi^1, \dots, \phi^m) \mid F \in C_0^1(\mathbb{R}^n)\}$$

is dense in \mathcal{C} with respect to \mathcal{E}_1 (the inner product from (DF2)).

Notice, that the assumption that $C^1(\Phi)$ is dense in \mathcal{C} implies that it is dense in $C_0(K)$ which means that it is point separating, so that it is an embedding into the space.

We shall refer to Φ as a function from $K \rightarrow \mathbb{R}^m$. We shall define $K_\Phi = \Phi(K)$ and $\tilde{\mu} = \Phi^* \mu$, where μ is a σ -finite Borel regular measure with full support. Because Φ is point separating, it is a homeomorphism between K and K_Φ .

The energy measures Γ satisfy the following chain rule from theorem 3.2.2 in [FOT11]: if for $f, h, g_1, \dots, g_k \in \mathcal{F}$ and $F \in C^1(\mathbb{R}^k)$, if $f = F(g_1, \dots, g_k)$, then

$$\Gamma(f, h) = \sum_{i=1}^k \frac{\partial F}{\partial x^i} \Gamma(g_i, h).$$

Proposition 6.1. *Elements of the form $\sum_{i=1}^m \phi^i \otimes \omega_i$ for $\omega_i \in \mathcal{C}$ are dense in \mathcal{H} .*

Proof. Using the chain rule for Γ one can show that

$$F(\phi^{i_1}, \dots, \phi^{i_n}) \otimes \omega = \sum_{k=1}^n \phi^{i_k} \otimes \left(\omega \frac{\partial F}{\partial x^{i_k}} \right)$$

and thus result then follows because $C^1(\Phi)$ is dense in \mathcal{F} . \square

Next, we define the map between differential forms on U to \mathcal{H} .

Definition 6.1. Define, for $\omega = \sum_{i=1}^m \omega_i dx^i$, the map $\pi : \Omega_C^1(U) \rightarrow \mathcal{H}$, by

$$\pi\omega = \sum_{i=1}^m \phi^i \otimes (\omega_i \circ \Phi)$$

and define the semi-norm $\|\omega\|_Z = \|\pi\omega\|_{\mathcal{H}}$. Define

$$\mathcal{N} = \Omega_C^1(U) / \ker \pi \cong \pi(\Omega_C^1(U))$$

Elements of the form $\sum_{i=1}^m \phi^i \otimes \omega_i = \pi\Omega_C^1(U)$ are dense in \mathcal{H} from proposition 6.1.

Proposition 6.2. *The seminorm $\|\cdot\|_Z$ has the following formula,*

$$\|\omega\|_Z^2 = \int_K \tilde{\omega}_i Z^{ij} \tilde{\omega}_j \, d\mu(x)$$

where $\tilde{\omega}_i = \omega_i \circ \Phi$ and

$$Z_x^{ij} = \frac{d\Gamma(\phi^i, \phi^j)}{d\mu}(x).$$

Thus $\|df\|_Z^2 = \mathcal{E}(f, f)$ and $d\mathcal{I}_{K_\Phi} \subset \ker \pi$. Further, if P_x is the orthogonal projection $T_{\Phi(x)}^* U \rightarrow T_{\Phi(x)} K_\Phi$, then $P_x Z_x P_x = Z_x$ for μ -almost every x .

Proof. To see the formula, let $\omega = \sum_{i=1}^n \omega_i dx^i$ and $\tilde{\omega}_i = \omega_i \circ \Phi$,

$$\langle \pi\omega, \pi\omega \rangle_\pi = \sum_{i,j} \langle \phi^i \otimes \tilde{\omega}_i, \phi^j \otimes \tilde{\omega}_j \rangle = \sum_{i,j} \int \omega_i Z^{ij} \omega_j \, d\mu.$$

Thus, by the chain rule,

$$\|df\|_Z = \sum_{i,j} \int \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} d\Gamma(\phi^i, \phi^j) = \mathcal{E}(f \circ \Phi, f \circ \Phi).$$

For any function $f \in \mathcal{I}_{K_\Phi}$, $f \circ \Phi$ is a constant. Thus $\|\pi df\|_{\mathcal{H}} = \mathcal{E}(f \circ \Phi) = 0$.

This implies that $P\omega = \sum_{i=1}^m P_x \omega_i dx^i$ is the fibrewise projection from $\Omega_C^1(U)$ to $\Omega_C^1(K)$, then $\langle P\eta, P\omega \rangle_Z = \langle \eta, \omega \rangle_Z$ for any $\eta, \omega \in \Omega_C^1(U)$. Since $\int_K \tilde{\omega}_i (PZP)^{ij} \tilde{\eta}_j \, d\mu = \int_K \tilde{\omega}_i Z^{ij} \tilde{\eta}_j \, d\mu$, it implies that $(P_x Z_x P_x)^{ij} = Z_x^{ij}$ for almost all x \square

Theorem 6.3.

- (1) π embeds \mathcal{N} into a dense subspace of \mathcal{H} .
- (2) Because $\Omega_C^1(K_\Phi) = \Omega_C^1(U) / d\mathcal{I}_{K_\Phi}$, and $d\mathcal{I}_{K_\Phi} \subset \ker \pi$, π descends to a homomorphism $\tilde{\pi} : \Omega_C^1(K_\Phi) \rightarrow \mathcal{H}$, which is given by the formula $\sum_i \omega_i d^K x^i \mapsto \sum_i \phi^i \otimes (\omega_i \circ \Phi)$.

(3) As a function from $\Omega_{L^2}^1(K) \rightarrow \mathcal{H}$, π is a densely defined closable operator and π^* is given by

$$(1) \quad \pi^* \sum_{i=1}^m \phi^i \otimes \omega_i = \sum_{i,j}^m Z_*^{ij} \omega_i d^K x^j.$$

where $Z_*^{ij}(x) = Z_y^{ij}$ if $x = \Phi(y)$.

Proof. Part (1) is a result of the fact that the image of π are tensors of the form $\sum_{i=1}^n \phi^i \otimes \omega_i$ for $\omega_i \in C(K)$, which are dense in \mathcal{H} by proposition 6.1. (2) is a corollary of proposition 6.2 and the homomorphism theorems.

(3) Because $\Omega_C^1(U)$ is dense in $\Omega_{L^2}^1(U)$, π is a densely defined operator there. Because elements of the form $\sum_{i=1}^m \phi^i \otimes \omega_i$, $\omega_i \in \mathcal{C}$ are dense in \mathcal{H} , formula (1) implies π^* is densely defined, and hence π is closable.

To see the formula,

$$\begin{aligned} \left\langle \pi \sum_{i=1}^m \eta_i dx^i, \sum_{i=1}^m \phi^i \otimes \omega_i \right\rangle &= \sum_{i,j=1}^m \int \tilde{\eta}_j Z^{ij} \omega_i d\mu(x) \\ &= \sum_{i,j=1}^m \int \tilde{\eta}_j (PZP)^{ij} \omega_i d\mu(x) \\ &= \left\langle \sum_{i=1}^m \eta_i dx^i, \pi^* \sum_{i=1}^m \phi^i \otimes \omega_i \right\rangle. \end{aligned}$$

□

We define $Z : \Omega_{L^2}^1(K) \rightarrow \Omega_{L^2}^1(K)$ by $Z := \pi^* \pi$

Lemma 6.4. For all $f, g \in C_0^1(K)$, then

$$\langle d_p^K g, (Zd^K f)_p \rangle_{K,p} = \frac{d\Gamma(f \circ \Phi, g \circ \Phi)}{d\mu}(p) = \sum_{i,j=1}^m Z_p^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}$$

for almost all p . In particular, since

$$\langle d_p^K f, (Zd^K f)_p \rangle_{K,p} = \frac{d\Gamma(f)}{d\mu}(p) \geq 0 \quad a.e.,$$

Z acts on almost every fiber of $\Omega_{L^2}^1(K)$ by the matrix Z_p^{ij} which is non-negative definite.

Give \mathcal{H} a \mathcal{C} -module structure on simple tensors with right action for $a, b, c \in \mathcal{C}$

$$a(b \otimes c) := b \otimes (ca)$$

Remark 6.5. This differs slightly from the bimodule structure which is typically given to \mathcal{H} . There is no loss in generality here because we are assuming that \mathcal{E} is local, and hence the chain rule for energy measures implies the left and the right actions coincide.

$\Omega_{C_1}^1(K)$ is a $C(K)$ -module with fiber-wise scalar multiplication: for $\phi \in C(K)$ and $\omega \in \Omega_{C_1}^1(K)$, $(\phi \cdot \omega)_p = \phi(p)\omega_p$. Then π is a \mathcal{C} -module homomorphism, that is $\pi(\phi \cdot \omega) = \phi \cdot \pi\omega$ for all $\omega \in \Omega_{C_1}^1(K)$, and $\phi \in \mathcal{C}$.

Because $\Gamma(\phi^i) \leq c_i \mu$ for some constant c_i , we have that there is a constant c such that $c \|\omega\|_{\Omega^1_{L^2}(K)} \geq \|\pi\omega\|_{\mathcal{H}}$.

Consider the metric

$$\rho_\mu(x, y) = \sup \{f(x) - f(y) \mid \|d\Gamma(f)/d\mu\|_\infty \leq 1\}$$

on the space K . This is called the intrinsic metric of on K with respect to the Dirichlet form as in, for example, [Stu94].

Proposition 6.6. *If Φ is a coordinate sequence as above, and Z is the related matrix as above*

$$\rho_\mu(x, y) = \sup \left\{ F \circ \Phi(x) - F \circ \Phi(y) \mid F \in C^1_0(U), \left\| \langle d^K F, Z d^K F \rangle_{K,p} \right\|_\infty \leq 1 \right\}.$$

In the rest of this section it is assumed that ρ_μ induces the original topology on K . This assumption and the fact that Φ is a coordinate sequence for μ , implies, by [HKT15], ρ_μ forms a shortest path (geodesic) distance on K . From corollary 6.4, we get that the following definition is equivalent to the above. That is if $\gamma : [a, b] \rightarrow K$, we define its length

$$L_\mu(\gamma) = \sup \left\{ \sum_{i=1}^k \rho_\mu(\gamma(t_{i-1}), \gamma(t_i)) \mid a = t_0 < t_1 < \dots < t_m = b \right\},$$

then $\rho_\mu(x, y) = \inf \{L_\mu(\gamma) \mid \gamma : [a, b] \rightarrow K, \gamma(a) = x, \gamma(b) = y\}$.

If we define $\lambda_m(Z_x)$ to be the largest eigenvalue of Z_x , then we have the following lemma.

Lemma 6.7. *There is a constant c_Z depending only on Φ such that*

$$\langle d_p^K F, (Z d^K F)_p \rangle_{K,p} \leq c_Z \|d_p^K F\|_{T_p^* K}^2 \leq c_Z |\nabla F(p)|$$

almost everywhere, where $c_Z \leq \|\lambda_m(Z_x)\|_\infty \leq \|\text{Tr } Z_x\|_\infty$.

Theorem 6.1. *If $\gamma : [a, b] \rightarrow K$ is a curve such that $L_\mu(\gamma) < \infty$, then $\tilde{\gamma} := \Phi \circ \gamma$ is a rectifiable curve in \mathbb{R}^m , and the Euclidean length of γ is bounded by $c_Z L_\mu(\gamma)$.*

Proof. Because of Lemma 6.7, assuming that γ is a unit-speed parametrization (i.e. $\rho_\mu(\gamma(t), \gamma(s)) = |t - s|$, which exists by standard metric space theory, see [BBI01] for example.),

$$|F(\tilde{\gamma}(t)) - F(\tilde{\gamma}(s))| \leq c_Z \rho_\mu(\gamma(t), \gamma(s)) \sup_{\tau \in [t, s]} |\nabla F(\tau)| = c_Z |t - s| \sup_{\tau \in [t, s]} |\nabla F(\tau)|$$

Thus $F \circ \tilde{\gamma}$ is a Lipschitz function for all F with bounded gradients. Thus by [Fed69], γ is almost everywhere differentiable, and $|d\tilde{\gamma}/dt| < c_Z$ almost everywhere. \square

6.1. Harmonic Coordinates on the Sierpinski Gasket. Let SG be the Sierpinski gasket with resistance form \mathcal{E} as in [Kig01, Str06], and let $\Phi = (\phi_1, \phi_2)$ be harmonic coordinates as in [Kig93, Kaj12, Kaj13]. In [Kig93], it is shown that for $f, g \in C^1(\mathbb{R})$ with $f - g \in \mathcal{I}_{\text{SG}_\Phi}$ if and only if $\nabla f = \nabla g$ on SG_Φ . This implies that $T_x^* \text{SG}_\Phi = T_x^* \mathbb{R}^2$ for $x \in \text{SG}_\Phi$. With this in mind, in this section we refer to d^{SG_Φ} as simply d . Further, it is proven in [Kig93] that Φ is a coordinate sequence.

We shall consider the space $\Omega_{L^2}^1(\text{SG}_\Phi, \nu)$ where $\nu = \Gamma(\phi^1) + \Gamma(\phi^2)$ is the Kusuoka measure. With the natural norm

$$\langle \omega_1 dx^1 + \omega_2 dx^2, \eta_1 dx^1 + \eta_2 dx^2 \rangle = \int \omega_1 \eta_1 + \omega_2 \eta_2 \, d\nu$$

In [Kig93] it is shown that there is a tensor field Z_x such that

$$\int \langle df(x), Z_x df(x) \rangle \, d\tilde{\nu}(x) = \mathcal{E}(f \circ \Phi) = \|\pi df\|_{\mathcal{H}}.$$

Since

$$\|\pi df\|_{\mathcal{H}} = \int \left| \frac{\partial f}{\partial x^1} \right|^2 d\Gamma(\phi_1) + \left| \frac{\partial f}{\partial x^2} \right|^2 d\Gamma(\phi_2) + 2 \frac{\partial f}{\partial x^1} \frac{\partial f}{\partial x^2} d\Gamma(\phi_1, \phi_2) = \mathcal{E}(f \circ \Phi),$$

we see that

$$(Z_x)_{ij} = \frac{d\Gamma(\phi^i, \phi^j)}{d\nu}.$$

This also implies that

$$(2) \quad \int g^2 \langle df(x), Z_x df(x) \rangle \, d\tilde{\nu}(x) = \|df\|_{\mathcal{H}} = \left\| \phi_1 \otimes \left(g \frac{\partial f}{\partial x^1} \right) + \phi_2 \otimes \left(g \frac{\partial f}{\partial x^2} \right) \right\|_{\mathcal{H}}$$

For the operator $\pi : \Omega_L^1(\text{SG}_\Phi) \rightarrow \mathcal{H}$

$$\pi(\omega_1 dx^1 + \omega_2 dx^2) = \phi^1 \otimes \omega_1 + \phi^2 \otimes \omega_2.$$

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